

First colonization of a hard-edge in random matrix theory

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Abstract

We describe the spectral statistics of the first *finite* number of eigenvalues in a newly-forming band on the hard-edge of the spectrum of a random Hermitean matrix model. It is found that in a suitable scaling regime, they are described by the same spectral statistics of a finite-size Laguerre-type matrix model. The method is rigorously based on the Riemann-Hilbert analysis of the corresponding orthogonal polynomials.

Keywords: Orthogonal polynomials, Random matrix theory, Schlesinger transformations, Riemann-Hilbert problems.

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1 Introduction

This paper is a sequel and a generalization of our previous paper [1]. We consider a Hermitean random matrix model where the spectrum of the matrix is constrained on a pre-determined union of intervals as well as being subject to a (varying) external potential. In this setting the endpoints of the intervals are called **hard-edges** and—typically—the density of eigenvalues near a hard-edge has power-law of the form $(x - x_0)^{-\frac{1}{2}}$. If—however—the external potential confines the asymptotic spectral density away from the hard-edge, the model is—de facto—independent of the location of the hard-edge.

In our previous paper we described the statistics of the first finitely many eigenvalues that start populating a newly-forming spectral band and showed that the predictions based on loop equations of [9] were in fact correct [16, 4]. Additionally we improved substantially the error estimate, allowing to treat the transitions occurring when the population increases by one. We called the point at which the spectral band is about to emerge the *spectral outpost*, picturing the eigenvalues as the first colonies of a large population (living within the main bands). Pushing the same analogy we could call the hard-edge a *port* which has a sea on one side where the colonies cannot live (i.e. where the eigenvalues are forbidden to be). The present paper is in the same spirit as [1], but we want to describe a similar first-colonization that occurs precisely at a hard-edge of the spectrum (or port). In the process we will improve some of the ingredients that appeared in [1].

In order to simplify the setup we will simply assume that the model we are describing consists of a model of *positive definite* Hermitean matrices $M \in \mathcal{H}_+$ (here \mathcal{H}_+ denotes the cone of positive definite Hermitean matrices), thereby putting the hard-edge at the origin. Of course one could as well consider any subset of \mathcal{H} determined by the requirement that $Sp(M) \subset \bigsqcup I_j$, i.e. the spectrum lies within a pre-determined union of disjoint intervals. Since our analysis is local to one hard-edge there is little loss of generality in restricting ourself to the case of positive matrices. The reader will recognize that all the considerations can be extended without much effort to the most general situation.

We thus consider the matrix model with (unnormalized) probability measure

$$d\mu := (\det M)^\alpha e^{-\frac{N}{T} Tr V(M)} dM, \quad \alpha > -1, \quad M > 0. \quad (1.1)$$

The potential $V(x)$ is a (scalar) function with the properties that it is *real-analytic* at $x = 0$ (i.e. at the hard-edge), it is bounded from below on \mathbb{R}_+ and grows faster than $\ln(1 + x^2)$ at infinity.

Following our idea in [1] we will use a simplified setup which dispenses us with the necessity of a complicated double-scaling limit that amounts to a step-wise (infinitesimal) modification of the potential near the hard-edge.

In order to explain the setup in more detail, suppose that the (electrostatic) effective potential [17] $\varphi(x) = \frac{1}{2}V(x) - g(x) + \frac{1}{2}\ell^3$ vanishes at the hard-edge $x = 0$ as $C_0 x^\nu$ with some $C_0 > 0$ (the positivity of C_0 means that the main spectral band is confined away from the hard-edge). We then modify the potential (Sect. 3.1) by adding a step-like perturbation of the form

$$V(x) \rightarrow \tilde{V}(x) = V(x) - \left(2T\gamma\kappa \frac{\ln N}{N} + \epsilon_N(x) \right) \chi_J(x), \quad \gamma := \frac{1}{\nu}. \quad (1.2)$$

Here χ_J is the characteristic function of a small interval J around the hard-edge. The discontinuous character of the perturbation makes no trouble whatsoever, since the characteristic function could be replaced by any smooth function that is identically 1 on J and identically vanishing in a slightly larger interval and all the analysis would be identical. The real parameter κ determines the strength of the perturbation and the constants are crafted for later convenience. The function $\epsilon_N(x)$, which is independent of κ and of order $N^{-\gamma}$, will be specified in due course.

Due to Dyson’s theorem, instead of studying directly the spectral statistics we focus on the associated orthogonal polynomials and the corresponding *Christoffel–Darboux kernel*, in terms of which all correlation functions of the eigenvalues can be written [15]. These will be studied with the Deift–Zhou steepest descent method [6] based on the formulation of the relevant Riemann–Hilbert problem as in [13].

For $\kappa < 0$ the orthogonal polynomials do not exhibit any peculiar behavior in the large N limit; for positive values of κ new zeroes of the OPs start appearing near the hard edge. The same transitional phenomenon as κ crosses a half-integer is observed naturally, here as in [1]: specifically the normalizations we have chosen are such that in the asymptotic regime there are K roots near the hard-edge, where K is the integer nearest to κ . Clearly transitions must occur at $\kappa \in \mathbb{N} + \frac{1}{2}$.

At the level of the random–matrix model it will appear that the first K eigenvalues that are growing from the hard-edge start populating the new band subject to the statistic of a **finite-size** $K \times K$ matrix model of positive matrices subject to the “microscopic” (denoted by the subscript “m” hereafter) measure

$$d\mu_m := \det(M_m)^{\alpha} e^{-\text{Tr } V_m(M_m)} \quad (1.3)$$

where M_m is a $K \times K$ positive definite Hermitean matrix (microscopic compared to the $N \rightarrow \infty$ original model) and $V_m(\zeta)$ will be an arbitrary monic polynomial of degree ν in ζ —a microscopic coordinate.

Although the main results will be along the same line of [1] there are a few interesting differences.

- The microscopic potential $V_m(x)$ can be directly controlled by $\epsilon_N(x)$ (1.2). In [1] $V_m(x)$ was only a monomial.
- The global parametrix requires a new piece called *Szegő function*. It can be written in an arbitrary higher genus using Theta functions.
- The method used in [1] of the “partial Schlesinger transform” was not completely general but seemed to require some (upper-triangular) structure of the local parametrix. Here we show that the same method can be applied in the most generic situation.
- The above method provides us with a recursive procedure to construct the asymptotic solution up to an arbitrary order of accuracy limited only by the error terms coming from the simple end-points of the spectrum.

In the main body of the paper we make the simplifying assumption that the support of the equilibrium measure consists of one interval (**one-cut assumption**) confined to the interior of the positive real axis. Of course, this assumption is not crucial to the main result. In Appendix C we show (in a somewhat sketchy form) how to generalize to an arbitrary number of cuts.

³The “ g -function” is a rather common object in this area of research and will be explained in due course.

2 General setting

2.1 Riemann-Hilbert problem of orthogonal polynomial

The setting of the present paper is basically identical to [6], [3], and especially to [1].

Consider a Hermitean matrix model with measure given by

$$\frac{1}{Z_N} (\det M)^\alpha e^{-\frac{N}{T} \text{tr} V(M)} dM, \quad (2.1)$$

where Z_N is the normalization constant. The potential $V(x)$ is assumed to be **real-analytic** on the half-line $x \geq 0$.

In particular $V(x)$ is analytic at $x = 0$.

Let $\{p_n(x) | n = 0, 1, 2, \dots\}$ be the corresponding (monic) OPs that satisfy the following orthogonality

$$\int_{\mathbb{R}_+} p_n(x) p_m(x) x^\alpha e^{-\frac{N}{T} V(x)} dx = h_n \delta_{nm}. \quad (2.2)$$

The spectral statistics of the model can be computed in terms of the Christoffel-Darboux kernel [15]

$$K(x, x') = \sum_{j=0}^{n-1} \frac{p_j(x) p_j(x')}{h_j} = \frac{p_n(x) p_{n-1}(x') - p_{n-1}(x) p_n(x')}{h_{n-1}(x - x')} \quad (2.3)$$

The asymptotic analysis hinges on the following characterization for the OPs in terms of the Riemann-Hilbert Problem (RHP) described hereafter. Define for $z \in \mathbb{C} \setminus \mathbb{R}_+$ the matrix

$$Y(z) := Y_n(z) := \begin{bmatrix} p_n(z) & \mathcal{C}[p_n](z) \\ \frac{-2i\pi}{h_{n-1}} p_{n-1}(z) & \frac{-2i\pi}{h_{n-1}} \mathcal{C}[p_{n-1}](z) \end{bmatrix}, \quad \mathcal{C}[p](z) := \frac{1}{2i\pi} \int_{\mathbb{R}_+} \frac{p(x) x^\alpha e^{-\frac{N}{T} V(x)} dx}{x - z}. \quad (2.4)$$

Here and below we use the coordinate z for the complex plane and x when restricted to the real line.

The above matrix has the following jump-relations and asymptotic behavior that uniquely characterize it [13, 14, 11, 12] (we drop the explicit dependence on n for brevity)

$$Y_+(x) = Y_-(x) \begin{bmatrix} 1 & x^\alpha e^{-\frac{N}{T} V(x)} \\ 0 & 1 \end{bmatrix} \text{ on } x \geq 0, \quad Y(z) \sim (\mathbf{1} + \mathcal{O}(z^{-1})) \begin{bmatrix} z^n & 0 \\ 0 & z^{-n} \end{bmatrix}. \quad (2.5)$$

Replacing the orthogonality condition (2.2) by the above jump (and boundary) conditions (2.5) we obtain the Riemann-Hilbert problem for the OPs. Using this setup we especially want to investigate the asymptotics of the OPs as their degree $n := N + r$ goes to infinity while r being fixed to an integer.

2.2 g -function

The asymptotic analysis is based on the construction of the so-called g -function; we recall that, in the simple case where the contour of integration in (2.2) is the real axis the equilibrium measure is obtained from the solution of a variational problem for a functional over probability measures on the real axis, in the sense of potential theory [17]. Indeed define the **weighted electrostatic energy** [17]

$$\mathcal{F}[\mu] := \int_{\mathbb{R}_+} V(x) d\mu(x) + \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \ln \frac{1}{|x - x'|} d\mu(x) d\mu(x'), \quad (2.6)$$

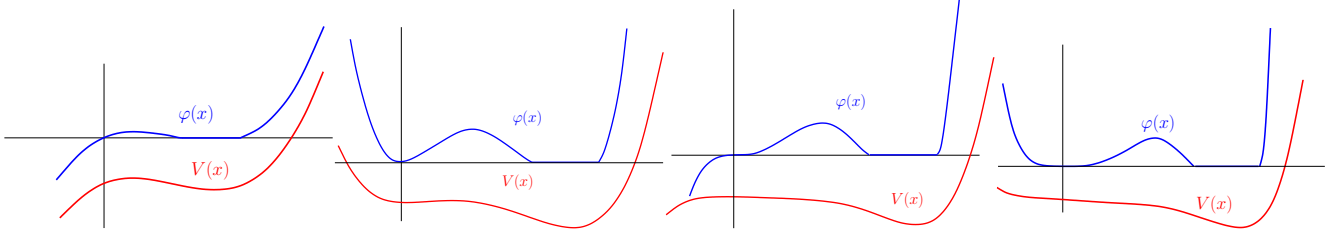


Figure 1: Examples of potentials V and corresponding effective potentials φ for critical hard-edge situations. The effective potential behaves as x^ν near the hard-edge, with $\nu = 1, 2, 3, 4$ respectively (from left to right). The plots are numerically accurate, but the vertical axes are scaled differently for V and φ .

where $d\mu$ is a positive measure supported on the positive real axis with total mass $T = \int_{\mathbb{R}_+} d\mu(x)$.

It is known that the functional \mathcal{F} attains a unique minimum (under mild assumptions on the growth of $V(x)$ at infinity) at a certain measure $\rho(x)$ that is called the **equilibrium measure** [17, 7].

It is also known [5] that the support of the measure $\rho(x)$ consists of a finite union of disjoint bounded intervals and that $\rho(x)$ is smooth on the interior of the support.

Taking avail of the equilibrium measure, the g -function [7] is defined as

$$g(z) := \int_{\mathbb{R}} \rho(x) \ln(z - x) dx = T \ln z + \mathcal{O}(z^{-1}), \quad (2.7)$$

where the logarithm must be defined with an appropriate cut extending—say—from the leftmost endpoint of the support of $\rho(x)$ to $+\infty$.

The main properties that enter the steepest descent analysis are the standard properties of the logarithmic transform. To this end we note that the representation (2.7) implies immediately that $\Re g(z)$ is harmonic away from the support of ρ and continuous on the whole complex plane. The Euler-Lagrange variational equations (equivalent to the optimality of the equilibrium measure ρ [17]) can be rephrased in terms of the following conditions for the g -function.

- for $x \in \mathbb{R}_+$ we have

$$\Re \varphi(x) \geq 0, \quad \varphi(z) := \frac{V(z)}{2} - g(z) + \frac{\ell}{2}, \quad (2.8)$$

for a suitable real constant ℓ . $\Re \varphi$ is the effective potential of the related electrostatic problem. Since the integration (the *conductor* in the terminology of [17]) terminates at the origin, the variational equations tell nothing on the sign of $\Re \varphi$ on the left of the hard-edge.

- The opposite inequality (and hence the equality) holds on the support of ρ . Especially, ℓ is chosen such that $\Re \varphi = 0$ on the support of ρ . (The support of ρ will be called the **cuts** because they form the cuts of the functions $g'(z)$ and $\varphi'(z)$.) Here and in the previous point, the g -function should be understood as the analytic function defined by its integral representation (2.7) on the simply connected domain obtained by removing a half-line starting e.g. at the leftmost endpoint of the support of ρ and extending towards ∞ . Then $\Re \varphi(x)$ is actually nothing but the boundary-value $\frac{1}{2}(\varphi_+(x) + \varphi_-(x))$ on the half-line.
- In suitable finite left/right neighborhoods of the cuts, the function $\Re \varphi(z)$, which is also harmonic in the domain of analyticity of $V(z)$, is **negative**.

We note that the effective potential is defined within the domain of analyticity of $V(z)$ and in particular in a left neighborhood of the hard-edge (the origin) due to our assumption that $V(z)$ is analytic at $z = 0$.

In particular $\Re\varphi$ is a real analytic function at $x = 0$ as long as the support of the equilibrium measure does not contain the hard-edge. If 0 belongs to the support then it is known [5] that—in general—the equilibrium measure $\rho(x)$ has an integrable singularity of type $1/\sqrt{x}$.

We consider the case where $0 \notin \text{supp}(\rho)$ but the inequality (2.8) is non strict and fails precisely at the hard-edge. Figure (2) shows the potential and the effective potential. Taking the cut of $g(z)$ to extend from the leftmost endpoints of the support of ρ to $+\infty$ we see that both $g(z)$ and $\varphi(z)$ are real-analytic functions in a finite neighborhood of the hard-edge $x = 0$. In particular, because of inequality (2.8) we must have

$$\varphi(x) = C_0 x^\nu (1 + \mathcal{O}(x)) , \quad C_0 > 0 , \quad \nu \in \mathbb{N}_+, \quad (2.9)$$

where ν can be any *positive* integer. The most generic situation is $\nu = 1$ when, as we will see, the microscopic ensemble reduces to a Laguerre ensemble.

2.3 Local coordinate around the hard-edge

At the hard-edge the effective potential behaves as $\varphi(z) = \frac{1}{2}V(z) - g(z) + \frac{1}{2}\ell \simeq C_0 z^\nu$ with $C_0 > 0$ (2.9). We define a new conformal parameter \tilde{z} as follows:

$$\tilde{z} := C_0^{-\gamma} \varphi(z)^\gamma = z + \mathcal{O}(z^2) =: z e^{\eta(z)}. \quad (2.10)$$

We define \mathbb{D} to be a finite open neighborhood around $z = 0$ that maps univalently to a *disk* centered at $\tilde{z} = 0$ by the above relation and such that $\mathbb{D} \cap \mathbb{R}_+ = [0, \epsilon)$ does not intersect the support of the equilibrium measure. We also define the **scaling conformal parameter** by

$$\zeta := \left(\frac{2N}{T} \varphi(z) \right)^\gamma = (\tilde{C}_0 N)^\gamma \tilde{z} \quad \text{where} \quad \tilde{C}_0 := \frac{2C_0}{T}. \quad (2.11)$$

The neighborhood \mathbb{D} maps to a disk centered at the origin in the ζ -plane whose diameter grows as N^γ .

We also define the function $\eta(z)$, which is holomorphic at the origin, by

$$e^{\eta(z)} := \left(\tilde{C}_0 N \right)^{-\gamma} \frac{\zeta(z)}{z} \quad \text{or} \quad \tilde{z} = z e^{\eta(z)}. \quad (2.12)$$

As can be seen $\eta(z)$ measures the mismatch (up to scaling) of the scaling conformal parameter ζ with respect to the original coordinate z .

3 Modifying the setting

3.1 Perturbation of the potential

Define $J = [0, \epsilon) = \mathbb{D} \cap \mathbb{R}_+$; consider then the following **modified orthogonality relation**:

$$h_{nm}\delta_{nm} = \int_{\mathbb{R}_+} p_n(x)p_m(x) x^\alpha e^{-\frac{N}{T}\tilde{V}(x)} dx, \quad (3.1)$$

where we have defined the **exponent of nonregularity** $\gamma := \frac{1}{\nu}$ and the **perturbed potential**

$$\tilde{V}(x) := V(x) + \frac{T}{N} \left(f(\zeta) - (2\kappa + \alpha)\gamma \ln N - \alpha\gamma \ln \tilde{C}_0 - \alpha\eta(x) \right) \chi_J(x). \quad (3.2)$$

The function $f(\zeta)$ is an arbitrarily chosen real polynomial of degree $\nu - 1$.

The combersome form of the perturbation is crafted so as to have the simplest local parametrix and so that the parameter $\kappa \in \mathbb{R}$ will eventually determine the size of the population of the colony near the hard-edge.

The polynomial $f(\zeta)$ is a **fine-tuning** which allows quite some additional generality: this term did not appear in [1] but one could re-read loc.cit. with this fine-tuning in place and the obvious changes while maintaining all the results valid.

Remark 3.1 For the cases $\nu \geq 2$ the fine-tuning $f(\zeta)$ is in fact quite a strong perturbation since it scales as $N^{1-\gamma}$ which is much stronger than the logarithmic perturbation.

3.2 Normalized and lens-opened RHP

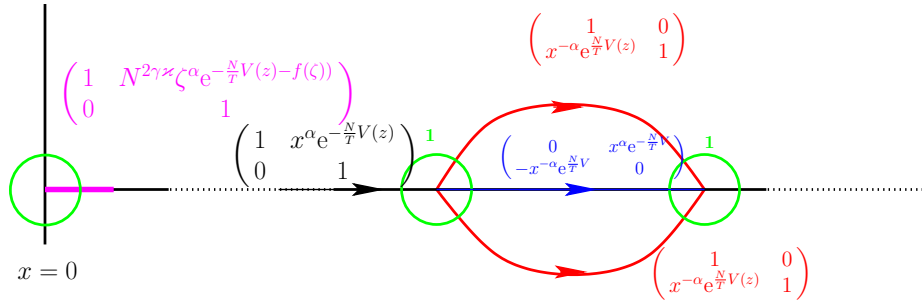


Figure 3: The jump matrices for Y .

identity but later we define separate RHP problems inside the circles. Then we will call all the RHP inside one of the green disks the **local problem** whereas we call the problem outside the **outer problem**.

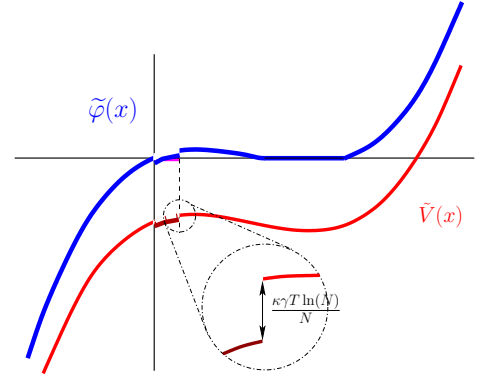


Figure 2: The potential \tilde{V} with the chemical potential added, and the corresponding effective potential. In this example $\nu = 1$ and the fine-tuning is thus absent.

In order to streamline the derivation we follow [1] and open the lenses **before** normalizing the problem, thus modifying the jumps as shown in figure 3. We thus redefine

$$Y_{\text{new}} := Y \begin{bmatrix} 1 & 0 \\ -x^{-\alpha} e^{\frac{N}{T}V(x)} & 1 \end{bmatrix}, \quad \text{on the upper lip,} \quad (3.3)$$

$$Y_{\text{new}} := Y \begin{bmatrix} 1 & 0 \\ x^{-\alpha} e^{\frac{N}{T}V(x)} & 1 \end{bmatrix}, \quad \text{on the lower lip.} \quad (3.4)$$

For the time being the jumps on the green circles are the identity

After the lens-opening we define

$$\tilde{Y}(z) := e^{-\frac{N\ell}{2T}\sigma_3} Y(z) e^{-\frac{N}{T}g(z)\sigma_3} e^{\frac{N\ell}{2T}\sigma_3}, \quad (3.5)$$

which satisfies a new RHP:

$$\tilde{Y}(x)_+ = \tilde{Y}(x)_- \begin{bmatrix} e^{\frac{N}{T}(g_- - g_+)} & x^\alpha e^{-\frac{N}{T}(V - g_+ - g_- + \ell)} \left(\tilde{C}_0^{\alpha\gamma} e^{-f(\zeta) + \alpha\eta(x)} N^{(2\kappa + \alpha)\gamma} \right)^{\chi_J(x)} \\ 0 & e^{\frac{N}{T}(g_+ - g_-)} \end{bmatrix} \quad (3.6)$$

$$\tilde{Y}(z) \simeq (\mathbf{1} + \mathcal{O}(z^{-1})) z^{T\sigma_3}, \quad z \sim \infty \quad (3.7)$$

For simplicity we assume that there is only one finite band in the spectrum. This means that $\text{supp}(\rho) = [a, b]$ and that the spectral curve $w^2 = (z-a)(z-b)$ is of genus 0; the generalization to more bands is not conceptually a problem but requires the use of Θ -functions which would make the note quite more technical and long. Under this assumption, for $x \in \mathbb{R}$,

$$g_+(x) = -g_-(x) + V(x) + \ell, \quad \text{for } x \in \mathbb{R}_+ \text{ on the cut}, \quad (3.8)$$

$$g_+(x) = g_-(x) - 2i\pi T, \quad \text{for } x \in \mathbb{R}_+ \text{ on the right of the cut}, \quad (3.9)$$

$$g_+(x) = g_-(x), \quad \text{for } x \in \mathbb{R}_+ \text{ on the left of the cut}. \quad (3.10)$$

Everywhere else on the complex plane $g(z)$ is holomorphic. On account of these properties for the g -function the jumps for \tilde{Y} are shown in the figure 4.

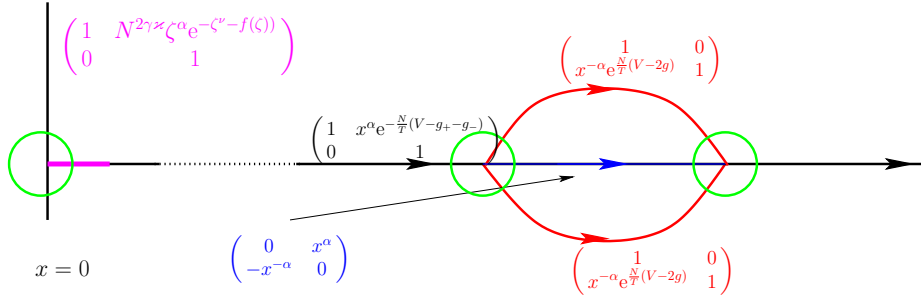


Figure 4: The jump matrices for \tilde{Y} .

In the following the size of the green circles will be fixed to a nonzero value. In this case the reader can verify that—*outside of the green disks*—the jumps on the black and red lines become exponentially close to the identity, and uniformly so in $L^2 \cap L^\infty$.

3.3 Outer parametrix: Part I

The variational equations (2.8 and following) imply that all the jumps on the red contours (the lips of the lenses) and on the spectral gaps are exponentially close to the identity jump; after removing them we are left with the jump matrices as shown in Figure 5. This provides the asymptotic RHP whose solution will be called the **outer parametrix**.

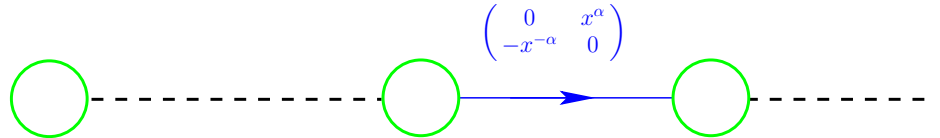


Figure 5: Jump matrix for Ψ_K .

The outer parametrix Ψ_{\varkappa} will be written in the form

$$\Psi_{\varkappa}(z) := F(z) \Psi_K(z), \quad (3.11)$$

where $F(z) = \mathbb{I} + F_{K,\delta}/z$ is a unimodular (i.e. $\det F(z) \equiv 1$) meromorphic function with a pole only at $z = 0$ (at the hard-edge); we will call this function a *partial Schlesinger transform* as in [1]. Remember that the parameter \varkappa appears in the potential (3.2) and we define the integer K as the nearest integer to \varkappa . Then we define a real number

$$\delta := \varkappa - K \quad \text{such that} \quad |\delta| \leq \frac{1}{2}. \quad (3.12)$$

The constant matrix $F_{K,\delta}$ will be obtained in section 3.6 after we introduce the local parametrix.

Ψ_K satisfies the following RHP.

$$\Psi_K(z) \simeq (\mathbf{1} + \mathcal{O}(z^{-1})) z^{r\sigma_3}, \quad z \sim \infty, \quad (3.13)$$

$$\Psi_K(x)_+ = \Psi_K(x)_- \begin{bmatrix} 0 & x^\alpha \\ -x^{-\alpha} & 0 \end{bmatrix}, \quad \text{on the cut.} \quad (3.14)$$

$$\Psi_K(z) = \mathcal{O}((z-a)^{-\frac{1}{4}}), \quad \Psi_K(z) = \mathcal{O}((z-b)^{-\frac{1}{4}}), \quad (3.15)$$

where a and b are the two turning points.

In fact, these conditions are still met with the left multiplication by $F(z)$, and we need an additional condition to define Ψ_K : which is the growth condition at the hard-edge.

$$\Psi_K(z) z^{-K\sigma_3} = \mathcal{O}(1). \quad (3.16)$$

The above four conditions (3.13)(3.14)(3.15)(3.16) give the Riemann-Hilbert problem for Ψ_K .

3.4 Outer parametrix: Part II (Szegő function)

Let t be the uniformizing map of the genus-0 Riemann surface. We let t_0 on the t -plane to map to the hard-edge on z -plane. For the simplicity of the normalization we choose the location of the cut and the hard-edge in the following way:

$$z(t) := \frac{b-a}{4} \left(t + \frac{1}{t} \right) + \frac{b+a}{2} = \frac{a-b}{4t_0} (t-t_0) \left(\frac{1}{t} - t_0 \right), \quad (3.17)$$

where $a < b$ are the endpoints of the band; in the t -plane they correspond to $t = \pm 1$. There are 2 points in the t -plane projecting to $z = 0$ namely the two solutions of $z(t) = 0$. We denote the one outside the unit circle by t_0 , the other being $\frac{1}{t_0}$.

In order to solve a RHP with x -dependent jumps (3.14) we find a function $D(x)$ (usually known as Szegő-function [18]) that satisfies

$$D_+(x)D_-(x) = x^\alpha, \text{ for } x \text{ on the cut(s)}, \quad (3.18)$$

$$\lim_{z \rightarrow \infty} D(z) = D_\infty \text{ where } D_\infty \in \mathbb{R}_+, \quad (3.19)$$

$$D(x) \text{ is analytic and non-zero otherwise.} \quad (3.20)$$

It is immediate to find that

$$\begin{aligned} D(x) &= \left(\frac{(t-1/t_0)}{t(t-t_0)} \right)^{\alpha/2} x^{\alpha/2} = \\ &= \left(\frac{b-a}{4} \right)^{\alpha/2} \left(\frac{t-1/t_0}{t} \right)^\alpha, \end{aligned} \quad (3.21)$$

where the cuts of the non-integer power go entirely in the “unphysical sheet” namely between $t = 0$ and $t = 1/t_0$. Also one can see that $D_\infty = \left(\frac{b-a}{4} \right)^{\alpha/2}$. In case of several cuts one needs to employ Theta functions and we will give the details in an appendix A.

We now describe a specific solution Ψ_K to the above RHP (3.13)(3.14)(3.15)(3.16).

We can write the following solution (a minor modification from [1] due to the presence of the Szegő function)

$$\Psi_K(z(t)) := (D_\infty)^{\sigma_3} t_0^{K\sigma_3} \frac{\left(\frac{b-a}{4} \right)^{r\sigma_3}}{\sqrt{\frac{4z'(t)}{b-a}}} \begin{bmatrix} t^r & \frac{i}{t^{r+1}} \\ -it^{r-1} & \frac{1}{t^r} \end{bmatrix} \left(\frac{t-t_0}{t_0t-1} \right)^{K\sigma_3} D^{-\sigma_3}. \quad (3.22)$$

Above, t lies outside the unit circle for z not on the cut (the **physical sheet**), using (3.17). We take the branch of the square root such that $\sqrt{\frac{4z'(t)}{b-a}} \sim 1$ near $t = \infty$.

3.5 The local parametrix at the hard-edge

Within the chosen neighborhood of the hard-edge (inside the green circle) the RHP for \tilde{Y} (3.5) reads

$$\begin{aligned} \tilde{Y}_+ &= \tilde{Y}_- \begin{bmatrix} 1 & z^\alpha \left(\frac{2C_0}{T} \right)^{\alpha\gamma} N^{(2\kappa+\alpha)\gamma} e^{\alpha\eta(z)} e^{-\frac{N}{T}(V(z)-2g(z)+\ell)-f(\zeta)} \\ 0 & 1 \end{bmatrix} = \tilde{Y}_- \begin{bmatrix} 1 & \zeta^\alpha N^{2\kappa\gamma} e^{-2\frac{N}{T}\varphi(z)-f(\zeta)} \\ 0 & 1 \end{bmatrix} = \\ &= \tilde{Y}_- \begin{bmatrix} 1 & \zeta^\alpha N^{2\kappa\gamma} e^{-V_m(\zeta)} \\ 0 & 1 \end{bmatrix} \quad \text{where } V_m(\zeta) := \zeta^\nu + f(\zeta). \end{aligned} \quad (3.23)$$

We demand that the local parametrix solve the exact same jump within \mathbb{D} . To define the local parametrix $R_\kappa(\zeta)$, let us first define $R_K(\zeta)$ that solves the following RHP:

$$R_K(\zeta)_+ = R_K(\zeta)_- \begin{bmatrix} 1 & \zeta^\alpha e^{-V_m(\zeta)} N^{2\kappa\gamma} \\ 0 & 1 \end{bmatrix} \quad \text{at } \zeta \in \mathbb{R}_+, \quad (3.24)$$

$$R_K(\zeta) \sim \zeta^{-K\sigma_3} \mathcal{O}(1) \quad \text{when } \zeta \rightarrow 0, \quad (3.25)$$

$$R_K(\zeta) \sim \mathbf{1} + \mathcal{O}(N^{-\epsilon}) \quad \text{for some } \epsilon > 0 \text{ on } z \in \partial\mathbb{D}. \quad (3.26)$$

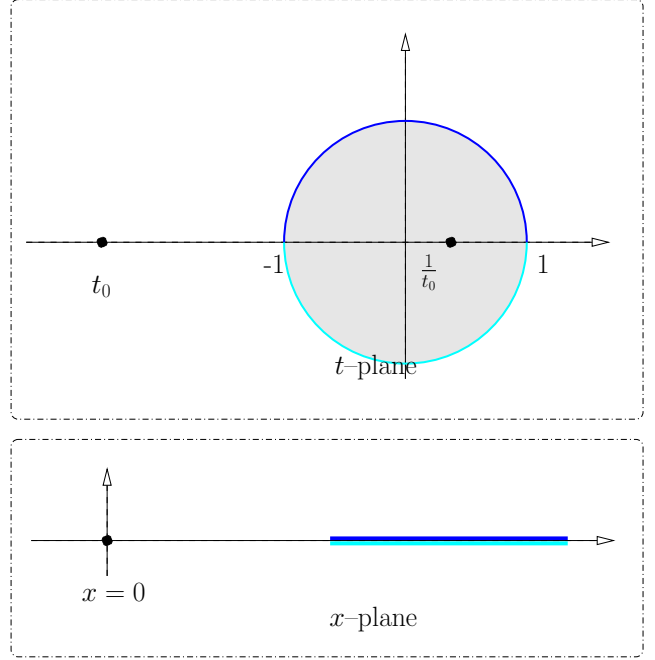


Figure 6: The uniformization of the plane sliced along the support of the equilibrium measure. The shaded region is the “unphysical sheet”.

Here ϵ is some positive number that will be determined in the subsequent analysis. Increasing ϵ leads to a better asymptotics. The second condition (3.25) is equivalent to demanding $\Psi_K R_K$ to be analytic, which implies that $\det R = 1$. Note that the singular behavior at the origin for the local parametrix is on the *left*.

Note the appearance of the **microscopic potential** V_m : in [1] this was completely determined by the effective potential and corresponds in the present case to setting the fine-tuning $f(\zeta)$ to zero. However we realize—a posteriori—that none of the analysis in [1] would be affected had we added the fine-tuning, as we are doing here.

3.5.1 Case $\varkappa \leq \frac{1}{2}$

If $\varkappa \leq \frac{1}{2}$ then the solution to the RHP (3.24)(3.25)(3.26) is written as

$$R_K(\zeta) := \begin{bmatrix} 1 & \frac{N^{2\varkappa\gamma}}{2i\pi} \int_{\mathbb{R}_+} \frac{\xi^\alpha e^{-V_m(\xi)} d\xi}{\xi - \zeta} \\ 0 & 1 \end{bmatrix} \quad \text{for } \varkappa \leq \frac{1}{2}. \quad (3.27)$$

For $\varkappa \leq 0$, we may define the local parametrix

$$R_\varkappa(\zeta) := R_K(\zeta) \quad \text{and} \quad F_{K,\delta} = 0 \quad \text{for } \varkappa \leq 0. \quad (3.28)$$

On $\partial\mathbb{D}$ we have

$$R_\varkappa(\zeta) \sim \mathbf{1} + \mathcal{O}(N^{2\gamma\varkappa-\gamma}). \quad (3.29)$$

As we will see, this becomes the error bound of our asymptotics.

When $\varkappa > 0$ the error term above will be larger than $\mathcal{O}(N^{-\gamma})$. We can lower the error bound using the partial Schlesinger transform as follows.

$$R_\varkappa(\zeta) := \begin{bmatrix} 1 & \frac{1}{\zeta} \frac{N^{2\varkappa\gamma}}{2i\pi} \int_{\mathbb{R}_+} \xi^\alpha e^{-V_m(\xi)} d\xi \\ 0 & 1 \end{bmatrix} R_{K=0}(\zeta) \sim \mathbf{1} + \mathcal{O}(N^{2\varkappa\gamma-2\gamma}) \quad \text{for } 0 < \varkappa \leq \frac{1}{2}. \quad (3.30)$$

3.5.2 Case $\varkappa \geq \frac{1}{2}$

We recognize in (3.24) the Riemann-Hilbert problem of the orthogonal polynomials for the weight $e^{-V_m(\zeta)} d\zeta$. Specifically, if we denote by $P_\ell(\zeta)$ the monic orthogonal polynomials that satisfy⁴

$$\int_{\mathbb{R}_+} P_\ell(\xi) P_{\ell'}(\xi) \xi^\alpha e^{-V_m(\xi)} d\xi = \eta_\ell \delta_{\ell\ell'} \quad \eta_\ell > 0, \quad (3.31)$$

then the solution of the RHP (3.24)(3.25)(3.26) is simply given by

$$R_K(\zeta) = \tilde{z}^{-K\sigma_3} H_K(\zeta), \quad (3.32)$$

where

$$H_K(\zeta) := \tilde{C}_0^{-\gamma K\sigma_3} N^{\gamma(\varkappa-K)\sigma_3} \begin{bmatrix} P_K(\zeta) & \frac{1}{2i\pi} \int_{\mathbb{R}_+} \frac{P_K(\xi) \xi^\alpha e^{-V_m(\xi)} d\xi}{\xi - \zeta} \\ \frac{-2i\pi}{\eta_{K-1}} P_{K-1}(\zeta) & \frac{-1}{\eta_{K-1}} \int_{\mathbb{R}_+} \frac{P_{K-1}(\xi) \xi^\alpha e^{-V_m(\xi)} d\xi}{\xi - \zeta} \end{bmatrix} N^{-K\gamma\sigma_3} N^{(K-\varkappa)\gamma\sigma_3}$$

⁴ In the simplest case $\nu = 1$ then $f(\zeta)$ is at most an irrelevant constant and the orthogonal polynomials used in the construction of the local parametrix are the Laguerre polynomials with parameter α .

$$= N^{\gamma\kappa\sigma_3} \left(\frac{\tilde{z}}{\zeta} \right)^{K\sigma_3} \left(\mathbb{I} + \mathcal{O} \left(\frac{1}{\zeta} \right) \right) \zeta^{K\sigma_3} N^{-\gamma\kappa\sigma_3}. \quad (3.33)$$

It is crucial to point out here that the right multiplier $N^{-\gamma\kappa\sigma_3}$ is needed to satisfy the correct jump relations (3.24), while the left multiplier $\tilde{C}_0^{-\gamma K\sigma_3} N^{\gamma(\kappa-K)\sigma_3}$ is needed to restore the boundary condition (3.25).

One can satisfy the boundary condition (3.26) only by choosing K as the closest integer of κ ; hence $\delta = \kappa - K \in (-\frac{1}{2}, \frac{1}{2})$ as we defined already in (3.12).

We obtain the following estimate holding **uniformly on the boundary**.

$$R_K = \mathbf{1} + \mathcal{O}(N^{-\gamma+2|\delta|\gamma}), \quad z \in \partial\mathbb{D}. \quad (3.34)$$

We see that if $\kappa \in \frac{1}{2} + \mathbb{Z}$ then the error term in (3.34) does not tend to zero (it is $\mathcal{O}(1)$). This is understandable as these values separate regimes where the value of K jumps by one unit and the whole strong asymptotic must changes its form. A similar problem arose in [9]. It was shown in [1] and [4] how to overcome the obstacle; there is nothing here that differs substantially from that case and the relevant analysis will be recalled later for the sake of completeness.

3.5.3 Improved local parametrix

To define the local parametrix R_κ in this case (and to find the appropriate $F(z)$ appearing in (3.11)), we first expand R_K for large ζ

$$R_K(\zeta) = \tilde{z}^{-K\sigma_3} \left(\mathbf{1} + \frac{1}{\zeta} \begin{bmatrix} a_K & -\frac{\eta_K}{2i\pi} \frac{N^{2\gamma\delta}}{\tilde{C}_0^{2\gamma K}} \\ -\frac{2i\pi}{\eta_{K-1}} \frac{\tilde{C}_0^{2\gamma K}}{N^{2\gamma\delta}} & -a_K \end{bmatrix} + \mathcal{O} \left(\frac{1}{\zeta^2} \right) \right) \tilde{z}^{K\sigma_3}, \quad (3.35)$$

where a_K is defined from $P_K^\nu(\zeta) = \zeta^\nu + a_K \zeta^{\nu-1} + \dots$.

Now we can define the local parametrix R_κ as follows.

$$R_\kappa(\zeta) := \tilde{z}^{-K\sigma_3} \left(\mathbf{1} - \frac{M_{K,\delta}}{\zeta} \right) H_K(\zeta) \quad \text{for } \kappa \geq \frac{1}{2}, \quad (3.36)$$

where the matrix $M_{K,\delta}$ is a *nilpotent* matrix defined as follows

$$M_{K,\delta} := \begin{cases} \begin{bmatrix} a_K & -\frac{\eta_K}{2i\pi} \frac{N^{2\gamma\delta}}{\tilde{C}_0^{2\gamma K}} \\ \frac{2i\pi}{\eta_K} a_K^2 \frac{\tilde{C}_0^{2\gamma K}}{N^{2\gamma\delta}} & -a_K \end{bmatrix} & \text{when } 0 < \delta \leq \frac{1}{2}, \\ \begin{bmatrix} a_K & \frac{\eta_{K-1}}{2i\pi} a_K^2 \frac{N^{2\gamma\delta}}{\tilde{C}_0^{2\gamma K}} \\ -\frac{2i\pi}{\eta_{K-1}} \frac{\tilde{C}_0^{2\gamma K}}{N^{2\gamma\delta}} & -a_K \end{bmatrix} & \text{when } -\frac{1}{2} \leq \delta < 0, \\ 0 & \text{when } \delta = 0. \end{cases} \quad (3.37)$$

These have been chosen such that R_κ gets as close as possible to the identity on the boundary $\partial\mathbb{D}$. We can verify that

$$R_\kappa = \mathbb{I} + \mathcal{O} \left(\frac{1}{N^{\gamma \min(1+2|\delta|, 2-2|\delta|)}} \right) \quad \text{on } \partial\mathbb{D}. \quad (3.38)$$

Inside the minimum function, $1+2|\delta|$ comes from the ζ^{-1} term and the $2-2|\delta|$ comes from the ζ^{-2} term.

3.6 Outer parametrix: Part III

As the last ingredient to the strong asymptotics, let us calculate $F(z) := \mathbf{1} + \frac{F_{K,\delta}}{z}$ (3.11) using the local parametrix obtained in the previous section. $F(z)$ is uniquely determined by the following analyticity condition at the outpost.

$$\mathcal{O}(1) = F(z) \Psi_K \tilde{z}^{-K\sigma_3} \left(\mathbf{1} - \frac{M_{K,\delta}}{\zeta} \right) = \left(\mathbf{1} + \frac{F_{K,\delta}}{z} \right) \Psi_K \tilde{z}^{-K\sigma_3} \left(\mathbf{1} - \frac{M_{K,\delta}}{(\tilde{C}_0 N)^\gamma} \frac{e^{-\eta(z)}}{z} \right). \quad (3.39)$$

This condition guarantees the analyticity of $\Psi_\varkappa R_\varkappa$ at the outpost.

The above condition is equivalent to the two equations below:

$$z^{-2} \text{ term : } 0 = F_{K,\delta} [\mathbf{A}_K^{(0)}, \mathbf{B}_K^{(0)}] M_{K,\delta}, \quad (3.40)$$

$$z^{-1} \text{ term : } 0 = F_{K,\delta} [\mathbf{A}_K^{(0)}, \mathbf{B}_K^{(0)}] - [\mathbf{A}_K^{(0)}, \mathbf{B}_K^{(0)}] \frac{M_{K,\delta}}{(\tilde{C}_0 N)^\gamma} - F_{K,\delta} [\mathbf{A}_K^{(1)}, \mathbf{B}_K^{(1)}] \frac{M_{K,\delta}}{(\tilde{C}_0 N)^\gamma}, \quad (3.41)$$

where the (matrix) coefficients $[\mathbf{A}_K^{(j)}, \mathbf{B}_K^{(j)}]$ are defined by the expansion below.

$$\Psi_K \tilde{z}^{-K\sigma_3} \left(\mathbf{1} - \frac{M_{K,\delta}}{(\tilde{C}_0 N)^\gamma} \frac{e^{-\eta(z)} - 1}{z} \right) =: [\mathbf{A}_K^{(0)}, \mathbf{B}_K^{(0)}] + [\mathbf{A}_K^{(1)}, \mathbf{B}_K^{(1)}]z + [\mathbf{A}_K^{(2)}, \mathbf{B}_K^{(2)}]z^2 + \dots \quad (3.42)$$

Each matrix $[\mathbf{A}_K^{(j)}, \mathbf{B}_K^{(j)}]$ for $j = 0, 1, \dots$, depends on N and has a (finite) limit at $N \rightarrow \infty$.

In fact, the equation (3.40) is contained in the other one (3.41). Solving the latter equation for $F_{K,\delta}$ and using (3.22), we are ready to write down the outer parametrix Ψ_\varkappa .

$$\Psi_\varkappa(z) := F(z) \Psi_K(z) \quad \text{for } \varkappa \geq \frac{1}{2}, \quad (3.43)$$

with

$$F(z) = \mathbf{1} + \frac{F_{K,\delta}}{z} \quad \text{where} \quad F_{K,\delta} = [\mathbf{A}_K^{(0)}, \mathbf{B}_K^{(0)}] \frac{M_{K,\delta}}{(\tilde{C}_0 N)^\gamma} \left([\mathbf{A}_K^{(0)}, \mathbf{B}_K^{(0)}] - [\mathbf{A}_K^{(1)}, \mathbf{B}_K^{(1)}] \frac{M_{K,\delta}}{(\tilde{C}_0 N)^\gamma} \right)^{-1}. \quad (3.44)$$

From the behavior of $M_{K,\delta}$ (3.37), $F(z)$ is a bounded function of N .

3.7 Asymptotic solution for \tilde{Y} and error analysis

Collecting all the results we present the following asymptotic solution for \tilde{Y} (3.5).

$$\tilde{Y}_\infty := \begin{cases} \Psi_\varkappa & \text{outside } \mathbb{D}, \\ \Psi_\varkappa R_\varkappa & \text{inside } \mathbb{D}. \end{cases} \quad (3.45)$$

Though we also need separate parametrices around the edges of the cut, we will skip those because they are well known in the literature; for instance, [6] (or our previous [1]).

To find the error term we define the *error matrix* as follows.

$$\mathcal{E}(z) := \tilde{Y}_\infty \tilde{Y}^{-1} \quad (3.46)$$

The error matrix solves the residual RHP with the jump matrices as shown in the figure 7. It follows from the construction that there is no jump inside the green disks, and on the cut.

On the disks around the edges of the cut (with airy parametrix), the jumps converge to the identity with uniform error bounds in $L^2 \cap L^\infty$ as (we simply cite the result, for instance from [6])

$$\mathcal{E}_+ \mathcal{E}_-^{-1} = \mathbf{1} + \mathcal{O}(N^{-1}). \quad (3.47)$$

The only non-trivial part is the jump on $\partial\mathbb{D}$ which is given by

$$\mathcal{E}_+(\mathcal{E}_-)^{-1} = \Psi_\varkappa R_\varkappa \Psi_\varkappa^{-1} = \mathbf{1} + \mathcal{O}\left(\frac{1}{N^\gamma \min(1+2|\delta|, 2-2|\delta|)}\right) \quad \text{on } \partial\mathbb{D}. \quad (3.48)$$

This is directly obtained from the error bound of R_\varkappa (3.38) because $\Psi_\varkappa(z)$ is uniformly bounded on $\partial\mathbb{D}$ for growing N (3.44).

A well-known theorem [7] guarantees that error matrix itself is bounded by the same error bound, i.e. $\mathcal{E} = \mathbf{1} + \mathcal{O}(N^{-\gamma-2\gamma|\delta|})$. This gives the following error term for the (strong) asymptotics of \tilde{Y} .

$$\tilde{Y} = \tilde{Y}_\infty \left(\mathbf{1} + \mathcal{O}\left(\frac{1}{N^\gamma \min(1+2|\delta|, 2-2|\delta|)}\right) \right). \quad (3.49)$$

4 Roots and kernel at the hard-edge

4.1 Case $|\varkappa - K| \neq \frac{1}{2}$

Using the strong asymptotics \tilde{Y}_∞ (3.45) *inside* the disk \mathbb{D} , let us look at the behavior of the roots at the hard-edge in detail.

The orthogonal polynomial appears at the (11)-component of the matrix Y . Due to the nature of the transformation: $Y \rightarrow \tilde{Y}$ (3.5) it is enough to consider \tilde{Y} to study the roots of that polynomial.

Collecting the previous results, let us write the full expression of \tilde{Y}_∞ to study the asymptotic behavior for large N .

$$\tilde{Y}_\infty = \Psi_\varkappa R_\varkappa = \left(\mathbf{1} + \frac{F_{K,\delta}}{z} \right) \Psi_K \tilde{z}^{-K\sigma_3} \left(\mathbf{1} - \frac{M_{K,\delta}}{\zeta} \right) \tilde{C}_0^{-\gamma K\sigma_3} \begin{bmatrix} P_K(\zeta) & * \\ \frac{-2i\pi}{\eta_{K-1} N^{2\gamma\delta}} P_{K-1}(\zeta) & * \end{bmatrix} N^{-K\gamma\sigma_3}. \quad (4.1)$$

The asterisks denote expressions which are irrelevant to the present analysis.

Defining a short-hand notation $\tilde{M}_{K,\delta} := (\tilde{C}_0 N)^{-\gamma} M_{K,\delta}$, which is of the order $\mathcal{O}(N^{2|\delta|\gamma-\gamma})$, we may evaluate the leading terms for large N as

$$\tilde{Y}_\infty = \left([\mathbf{A}_K^{(0)}, \mathbf{B}_K^{(0)}] + \underbrace{\mathcal{O}(N^{2|\delta|\gamma-\gamma})}_{\zeta \text{ independent}} + \mathcal{O}(N^{-\gamma}) \mathcal{O}(\zeta) \right) \tilde{C}_0^{-\gamma K\sigma_3} N^{-K\gamma} \begin{bmatrix} P_K(\zeta) & * \\ \frac{-2i\pi}{\eta_{K-1} N^{2\gamma\delta}} P_{K-1}(\zeta) & * \end{bmatrix}. \quad (4.2)$$

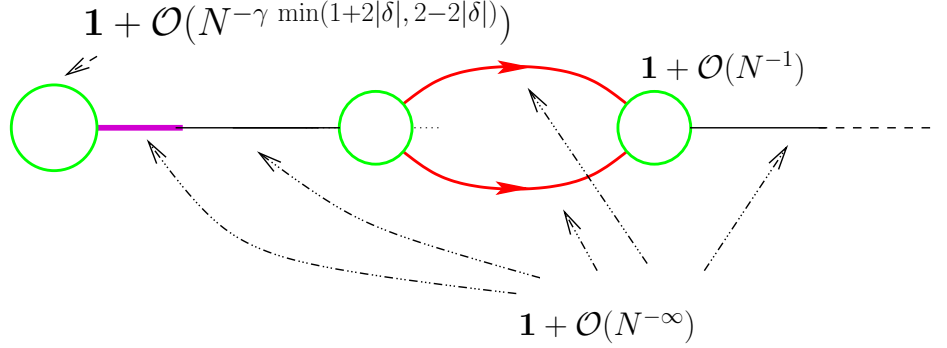


Figure 7: The jumps of \mathcal{E} .

The expression above implies that the microscopic zeroes are asymptotically close to those of $P_K(\zeta)$ if $\delta > 0$, or $P_{K-1}(\zeta)$ if $\delta < 0$ with one stray zero elsewhere. The details on the subleading behavior of the zeroes are basically equivalent to the description in [1], section 4.4. Here we summarize the scaling behavior of zeroes when \varkappa is not a half integer.

- $K < \varkappa < K + \frac{1}{2}$: All K zeroes are at a distance of order $\mathcal{O}(N^{-2\gamma\delta})$ (in the ζ -coordinate) from those of $P_K(\zeta)$.
- $K - \frac{1}{2} < \varkappa < K$. There are K zeroes in total; $K - 1$ zeroes are within a $\mathcal{O}(N^{-2\gamma|\delta|})$ distance from those of $P_{K-1}(\zeta)$ and the other zero is running away to the infinity at the rate $\sim N^{2\gamma|\delta|}$ in the ζ -coordinate. We may locate this zero as

$$\zeta_{\text{away}} \approx \frac{2i\pi\tilde{C}_0^{2\gamma K}\mathbf{B}_1^{(0)}}{\eta_{K-1}\mathbf{A}_1^{(0)}}N^{2\gamma|\delta|} \quad \text{where} \quad [\mathbf{A}_K^{(0)}, \mathbf{B}_K^{(0)}] = \begin{bmatrix} \mathbf{A}_1^{(0)} & \mathbf{B}_1^{(0)} \\ \mathbf{A}_2^{(0)} & \mathbf{B}_2^{(0)} \end{bmatrix}. \quad (4.3)$$

Using the explicit representation (3.22) we obtain

$$\zeta_{\text{away}} \approx -\frac{\pi\tilde{C}_0^{2\gamma K}2^{-2(2K+\alpha)+1}(b-a)^\alpha}{\eta_{K-1}t_0^{1+2r}}\left(1-\frac{1}{t_0}\right)^{2\alpha}\left(\frac{t_0^2}{(b-a)(1-t_0^2)^2}\right)^{-2K}N^{2\gamma|\delta|}. \quad (4.4)$$

Considering the relative location $a < b$ and $t_0 < 0$ (see Figure 6), the above quantity is positive, which means that the stray zero runs away to $\zeta = +\infty$ (namely towards the main spectral band). Note that in the x -coordinate the stray zero actually moves *towards* the hard-edge, but at a slower rate than the other ones, namely as $\mathcal{O}(N^{\gamma(2|\delta|-1)})$.

- $\varkappa = K$: There are K zeroes are at a distance of order $\mathcal{O}(N^{-\gamma})$ from the zeroes of $P_K(\zeta) - \frac{2i\pi\tilde{C}_0^{2\gamma K}\mathbf{B}_1^{(0)}}{\eta_{K-1}\mathbf{A}_1^{(0)}}P_{K-1}(\zeta)$.

The Christoffel–Darboux kernel (in terms of which all the spectral statistics can be computed) is given by

$$\begin{aligned} K_n(z, z') &:= \frac{1}{h_{n-1}} \frac{p_n(z)p_{n-1}(z') - p_{n-1}(z)p_n(z')}{z - z'} = \frac{1}{2i\pi} \frac{\det \begin{bmatrix} p_n(z) & p_n(z') \\ \frac{-2i\pi}{h_{n-1}}p_{n-1}(z) & \frac{-2i\pi}{h_{n-1}}p_{n-1}(z') \end{bmatrix}}{z' - z} = \\ &= \frac{e^{\frac{N}{T}(g(z)+g(z')-\ell)}}{2i\pi} \frac{1}{z' - z} \det \begin{bmatrix} \tilde{Y}_{11}(z) & \tilde{Y}_{11}(z') \\ \tilde{Y}_{21}(z) & \tilde{Y}_{21}(z') \end{bmatrix}. \end{aligned} \quad (4.5)$$

From the above and from (4.2) we can evaluate the leading behavior of the kernel in the scaling coordinate ζ . Using our asymptotics $\tilde{Y} \simeq \Psi_\varkappa R_\varkappa$, we evaluate the kernel in the local coordinate near the hard-edge as

$$K_n(\zeta, \zeta') \simeq \frac{\tilde{C}_0^\gamma e^{\frac{N}{T}(g(\zeta)+g(\zeta')-\ell)}}{N^{\gamma(2\varkappa-1)}} \frac{P_K(\zeta)P_{K-1}(\zeta') - P_K(\zeta')P_{K-1}(\zeta)}{\eta_{K-1}(\zeta - \zeta')} (1 + \mathcal{O}(N^{2\gamma|\delta|-\gamma})) \quad \text{when} \quad |\delta| \neq \frac{1}{2}. \quad (4.6)$$

This implies that the statistics of eigenvalues near the hard-edge in the scaling parameter ζ is governed by the microscopic (finite-size) matrix model of “Laguerre” type.

4.2 Case $|\varkappa - K| = \frac{1}{2}$

When \varkappa is a half-integer, i.e. when $|\delta| = \frac{1}{2}$, the ζ -independent error term (underbraced part) of (4.2) becomes $\mathcal{O}(1)$. The following fact is useful:

$$\frac{M_{K,\delta}}{(\tilde{C}_0 N)^\gamma} = \begin{cases} \begin{bmatrix} \mathcal{O}(N^{-\gamma}) & -\frac{\eta_K}{2i\pi} \tilde{C}_0^{-\gamma(2K+1)} \\ \mathcal{O}(N^{-2\gamma}) & \mathcal{O}(N^{-\gamma}) \end{bmatrix} = -u_K \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \mathcal{O}(N^{-\gamma}) & \text{when } \delta = \frac{1}{2}, \\ \begin{bmatrix} \mathcal{O}(N^{-\gamma}) & \mathcal{O}(N^{-2\gamma}) \\ -\frac{2i\pi}{\eta_{K-1}} \tilde{C}_0^{\gamma(2K-1)} & \mathcal{O}(N^{-\gamma}) \end{bmatrix} = -\ell_{K-1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \mathcal{O}(N^{-\gamma}) & \text{when } \delta = -\frac{1}{2}, \end{cases} \quad (4.7)$$

where we have defined (similar to (4.10) in [1])

$$u_K := \frac{\eta_K}{2i\pi} \tilde{C}_0^{-\gamma(2K+1)}, \quad \ell_{K-1} := \frac{2i\pi}{\eta_{K-1}} \tilde{C}_0^{\gamma(2K-1)}. \quad (4.8)$$

By explicitly writing the components as

$$[\mathbf{A}_K^{(j)}, \mathbf{B}_K^{(j)}] = \begin{bmatrix} \mathbf{A}_1^{(j)} & \mathbf{B}_1^{(j)} \\ \mathbf{A}_2^{(j)} & \mathbf{B}_2^{(j)} \end{bmatrix}, \quad (4.9)$$

the leading behavior of $F_{K,\pm\frac{1}{2}}$ (3.44) is written as

$$F_{K,\frac{1}{2}} = \frac{u_K}{1 + u_K \det[\mathbf{A}_K^{(0)}, \mathbf{A}_K^{(1)}]} \begin{bmatrix} \mathbf{A}_1^{(0)} & 0 \\ 0 & \mathbf{A}_2^{(0)} \end{bmatrix} \begin{bmatrix} \mathbf{A}_2^{(0)} & -\mathbf{A}_1^{(0)} \\ \mathbf{A}_2^{(0)} & -\mathbf{A}_1^{(0)} \end{bmatrix} + \mathcal{O}(N^{-\gamma}), \quad (4.10)$$

$$F_{K,-\frac{1}{2}} = -\frac{\ell_{K-1}}{1 + \ell_{K-1} \det[\mathbf{B}_K^{(1)}, \mathbf{B}_K^{(0)}]} \begin{bmatrix} \mathbf{B}_1^{(0)} & 0 \\ 0 & \mathbf{B}_2^{(0)} \end{bmatrix} \begin{bmatrix} \mathbf{B}_2^{(0)} & -\mathbf{B}_1^{(0)} \\ \mathbf{B}_2^{(0)} & -\mathbf{B}_1^{(0)} \end{bmatrix} + \mathcal{O}(N^{-\gamma}). \quad (4.11)$$

where the subscripts stand for the entries of the corresponding matrix.

A straightforward calculation gives the first column of \tilde{Y}_∞ as

$$\tilde{C}_0^{\gamma K} N^{\gamma K} \begin{bmatrix} (\tilde{Y}_\infty)_{11} \\ (\tilde{Y}_\infty)_{21} \end{bmatrix} \approx \frac{P_K(\zeta)}{1 + u_K \det[\mathbf{A}_K^{(0)}, \mathbf{A}_K^{(1)}]} \begin{bmatrix} \mathbf{A}_1^{(0)} \\ \mathbf{A}_2^{(0)} \end{bmatrix} \quad (4.12)$$

$$\begin{aligned} & -\frac{\ell_{K-1} \tilde{C}_0^\gamma}{N^\gamma} P_{K-1}(\zeta) \left(\begin{bmatrix} \mathbf{B}_1^{(0)} \\ \mathbf{B}_2^{(0)} \end{bmatrix} + u_K \begin{bmatrix} \mathbf{A}_1^{(1)} \\ \mathbf{A}_2^{(1)} \end{bmatrix} + \frac{u_K \det[\mathbf{B}_K^{(1)}, \mathbf{A}_K^{(0)}] + u_K^2 \det[\mathbf{A}_K^{(2)}, \mathbf{A}_K^{(0)}]}{1 + u_K \det[\mathbf{A}_K^{(0)}, \mathbf{A}_K^{(1)}]} \begin{bmatrix} \mathbf{A}_1^{(0)} \\ \mathbf{A}_2^{(0)} \end{bmatrix} \right) \\ & + \frac{\zeta P_K(\zeta)}{\tilde{C}_0^\gamma N^\gamma} \left(\begin{bmatrix} \mathbf{A}_1^{(1)} \\ \mathbf{A}_2^{(1)} \end{bmatrix} + \begin{bmatrix} \mathbf{A}_1^{(0)} \\ \mathbf{A}_2^{(0)} \end{bmatrix} \frac{u_K \det[\mathbf{A}_K^{(2)}, \mathbf{A}_K^{(0)}]}{1 + u_K \det[\mathbf{A}_K^{(0)}, \mathbf{A}_K^{(1)}]} \right) + \mathcal{O}(N^{-2\gamma}) \quad \text{for } \varkappa = K + \frac{1}{2}, \\ & \tilde{C}_0^{\gamma K} N^{\gamma K} \begin{bmatrix} (\tilde{Y}_\infty)_{11} \\ (\tilde{Y}_\infty)_{21} \end{bmatrix} \approx -\frac{\ell_{K-1} \tilde{C}_0^\gamma N^\gamma P_{K-1}(\zeta)}{1 + \ell_{K-1} \det[\mathbf{B}_K^{(1)}, \mathbf{B}_K^{(0)}]} \begin{bmatrix} \mathbf{B}_1^{(0)} \\ \mathbf{B}_2^{(0)} \end{bmatrix} \\ & + P_K(\zeta) \left(\begin{bmatrix} \mathbf{A}_1^{(0)} \\ \mathbf{A}_2^{(0)} \end{bmatrix} + \ell_{K-1} \begin{bmatrix} \mathbf{B}_1^{(1)} \\ \mathbf{B}_2^{(1)} \end{bmatrix} - \frac{\ell_{K-1} \det[\mathbf{A}_K^{(1)}, \mathbf{B}_K^{(0)}] + \ell_{K-1}^2 \det[\mathbf{B}_K^{(2)}, \mathbf{B}_K^{(0)}]}{1 + \ell_{K-1} \det[\mathbf{B}_K^{(1)}, \mathbf{B}_K^{(0)}]} \begin{bmatrix} \mathbf{B}_1^{(0)} \\ \mathbf{B}_2^{(0)} \end{bmatrix} \right) \\ & - \ell_{K-1} \zeta P_{K-1}(\zeta) \left(\begin{bmatrix} \mathbf{B}_1^{(1)} \\ \mathbf{B}_2^{(1)} \end{bmatrix} - \begin{bmatrix} \mathbf{B}_1^{(0)} \\ \mathbf{B}_2^{(0)} \end{bmatrix} \frac{\ell_{K-1} \det[\mathbf{B}_K^{(2)}, \mathbf{B}_K^{(0)}]}{1 + \ell_{K-1} \det[\mathbf{B}_K^{(1)}, \mathbf{B}_K^{(0)}]} \right) + \mathcal{O}(N^{-2\gamma}) \quad \text{for } \varkappa = K - \frac{1}{2}. \end{aligned} \quad (4.13)$$

Here, both approximations consist of three terms which—we must note—are each accurate up to multiplicative errors of $1 + \mathcal{O}(N^{-\gamma})$. A term containing other dependency on ζ —such as $\zeta^2 P_K(\zeta)$, for instance—comes with $\mathcal{O}(N^{-2\gamma})$ as noted in the formula.

The above expressions (4.12)(4.13) can tell the (asymptotic) location of K (or $K - 1$ for the latter formula) zeroes. However, these expressions are not enough to determine the stray zero; for this, we need *all the subleading terms*. Or, the location of stray zeroes have to be computed using the *exact* improved outer parametrix.

Finally, using (4.12)(4.13) we write the kernel at $\varkappa = K \pm \frac{1}{2}$.

$$K_n(\zeta, \zeta') \approx \frac{\tilde{C}_0^\gamma e^{\frac{N}{T}(g(z)+g(z')-\ell)}}{N^{2\gamma K}} \left(\frac{P_K(\zeta)P_{K-1}(\zeta') - P_K(\zeta')P_{K-1}(\zeta)}{\eta_{K-1}(\zeta - \zeta')} + \frac{\alpha_K}{\eta_K} P_K(\zeta)P_K(\zeta') \right) \quad (4.14)$$

$$\alpha_K := \frac{u_K \det[\mathbf{A}_K^{(0)}, \mathbf{A}_K^{(1)}]}{1 + u_K \det[\mathbf{A}_K^{(0)}, \mathbf{A}_K^{(1)}]} \quad \text{for } \varkappa = K + \frac{1}{2}, \quad (4.15)$$

$$K_n(\zeta, \zeta') \approx \frac{\tilde{C}_0^\gamma e^{\frac{N}{T}(g(z)+g(z')-\ell)}}{N^{2\gamma(K-1)}} \left(\frac{P_K(\zeta)P_{K-1}(\zeta') - P_K(\zeta')P_{K-1}(\zeta)}{\eta_{K-1}(\zeta - \zeta')} - \frac{\beta_{K-1}}{\eta_{K-1}} P_{K-1}(\zeta)P_{K-1}(\zeta') \right) \quad (4.16)$$

$$\beta_{K-1} := \frac{\ell_{K-1} \det[\mathbf{B}_K^{(1)}, \mathbf{B}_K^{(0)}]}{1 + \ell_{K-1} \det[\mathbf{B}_K^{(1)}, \mathbf{B}_K^{(0)}]} \quad \text{for } \varkappa = K - \frac{1}{2}. \quad (4.17)$$

These can be considered as “transitional” kernels when the size of the microscopic matrix model is a half-integer. Though these kernels do not seem to come from any simple microscopic matrix model, their structures suggest very natural interpretation: a mixture (sum) of two adjacent kernels. Such behavior of the kernel has been already observed in [1].

We conclude this section with the remark that the above analysis requires non-vanishing denominators $1 + u_K \det[\mathbf{A}_K^{(0)}, \mathbf{A}_K^{(1)}]$ and $1 + \ell_{K-1} \det[\mathbf{B}_K^{(1)}, \mathbf{B}_K^{(0)}]$. Using the explicit representation (3.22) we check (below) that the denominator does not vanish at least for genus 0.

$$u_K \det[\mathbf{A}_K^{(0)}, \mathbf{A}_K^{(1)}] = \frac{\eta_K 4^{1+2K+\alpha} t_0^{2r}}{2\pi \tilde{C}_0^{\gamma(2K+1)} (b-a)^\alpha} \left(1 - \frac{1}{t_0}\right)^{-2\alpha} \left(\frac{t_0^2}{(b-a)(1-t_0^2)^2}\right)^{1+2K} > 0 \quad \text{for } \varkappa = K + \frac{1}{2}, \quad (4.18)$$

$$\ell_{K-1} \det[\mathbf{B}_K^{(1)}, \mathbf{B}_K^{(0)}] = \frac{2\pi (b-a)^\alpha \tilde{C}_0^{\gamma(2K-1)}}{4^{-1+2K+\alpha} \eta_{K-1} t_0^{2r}} \left(1 - \frac{1}{t_0}\right)^{2\alpha} \left(\frac{t_0^2}{(b-a)(1-t_0^2)^2}\right)^{1-2K} > 0 \quad \text{for } \varkappa = K - \frac{1}{2}. \quad (4.19)$$

A Szegő function in arbitrary genus

We only sketch the construction since the details would require a good deal of notations to be set up. We will use the same notations and ideas contained in [2].

We denote by w double-cover of the z -plane branched at the endpoints of the support of the equilibrium measure

$$w^2 := \prod_{j=1}^{2g+2} (z - \alpha_j) \quad (A.1)$$

This is a hyperelliptic algebraic curve of genus g . We denote by ∞_\pm the two points above $z = \infty$ in the usual compactification of the curve, and by p_\pm the two points projecting to the location of the hard-edge⁵. We denote by ω_j the first-kind differentials normalized along the a -cycles: explicitly

$$\omega_j(z) = \sigma_{j\ell} \frac{z^{\ell-1} dz}{w}, \quad (A.2)$$

⁵The point ∞_+ is characterized by $w > 0$ as $z \in \mathbb{R}_+$ near ∞ . The point p_+ is the point on the Riemann surface of w obtained by analytic continuation of w on the complex plane slit along $(\alpha_{2j-1}, \alpha_{2j})$

where the summation over repeated indices is understood (and they range from 1 to g) and $\sigma_{j\ell}$ is an invertible matrix such that $\oint_{a_k} \omega_j = \delta_{jk}$.

The notation (rather standard) is lifted from [2] and [10]: the Abel map is understood when writing points as arguments of Θ and it is based at one of the Weierstrass points (for example α_1) Δ is an arbitrary odd non-singular half-period. Recall that (pag. 23 of [10]) all such characteristics Δ are in one-to-one correspondence with partitions of the Weierstrass points into $g-1$ and $g+3$ points $\{\alpha_{k_1}, \dots, \alpha_{k_{g-1}}\} \sqcup \{\alpha_{\bar{k}_1}, \dots, \alpha_{\bar{k}_{g+3}}\}$

Moreover, straightforward computations show that (with some *overall* ambiguity of signs)

$$\Theta_{\Delta}(p - \infty_{\pm}) \underset{p \rightarrow \infty_{\pm}}{\sim} \mp \frac{1}{z} \partial_{\ell} \Theta_{\Delta}(0) \sigma_{\ell g} \quad (\text{A.3})$$

The Szegő function we want to define now is a generalization of RHP (3.18)(3.19)(3.20). Let

$$v(z) := \prod_{j=1}^m (z - c_j)^{2\rho_j} \quad (\text{A.4})$$

where ρ_j are some real numbers and the points $c_j \in \mathbb{R}$ do not belong to the spectral bands. This function is defined on a simply connected domain obtained by removing (for example) vertical half-lines in the upper half-plane. The Szegő function is characterized by the following scalar Riemann–Hilbert problem

$$D_+(z)D_-(z) = v(z), \quad \text{for } z \text{ on the cut(s)}, \quad (\text{A.5})$$

$$\lim_{z \rightarrow \infty} D(z) = D_{\infty} \quad \text{where } D_{\infty} \in \mathbb{R}_+, \quad (\text{A.6})$$

$$D(x) \text{ is analytic and non-zero otherwise.} \quad (\text{A.7})$$

We denote by $\infty_{\pm}, \pm\gamma_j$ the points on the hyperelliptic curve above $z = \infty, z = c_j$ on the two sheets of the covering defined in the customary way; we will refer to the sheet of the points ∞_+, γ_j 's as the *physical sheet*. We leave it to the reader to verify that the solution can be written as follows

$$D(z) = \prod_{j=1}^m \left(\frac{\Theta_{\Delta}(p - \infty_-) \Theta_{\Delta}(p + \gamma_j)}{\Theta_{\Delta}(p - \infty_+) \Theta_{\Delta}(p - \gamma_j)} \right)^{\rho_j} v(z)^{1/2}. \quad (\text{A.8})$$

The first and third conditions (A.5)(A.7) are easily checked. The second condition (A.6) can be verified by taking the limit of the proposed expression

$$\lim_{z \rightarrow \infty} D(z) = \left(\frac{\Theta_{\Delta}(\infty_+ - \infty_-)}{\partial_{\ell} \Theta_{\Delta}(0) \sigma_{\ell g}} \right)^{\sum \rho_j} \prod_{j=1}^m \frac{\Theta_{\Delta}(\infty_+ + \gamma_j)}{\Theta_{\Delta}(\infty_+ - \gamma_j)} \quad (\text{A.9})$$

Note that -since all branchpoints are real- the Theta-function $\Theta_{\Delta}(p - \infty_+)$ is real for any $p \in \mathbb{R}$ and outside of the spectral bands as follows from ([10], Chapter VI, Prop. 6.1). So is the matrix σ_{kl} and the derivatives of Θ_{Δ} at the origin. Hence the expression above is real and -in case it is negative- we can simply change the overall sign to the expression for D .

The expression above needs to be supplemented by specifying where the cuts are made; for the denominator the cuts run on the physical sheet between γ_j and ∞_+ , while for the numerator they run similarly but on the second sheet. It is simple to verify (due to the fact that $\Theta_{\Delta}(p - q)$ has a simple zero for $p = q$) that in fact the analytic continuation of $D(x)$ around each c_j yields the same germ of holomorphic function and hence $D(x)$ has -in fact- no cuts on the physical sheet.

B Partial Schlesinger transform and any order asymptotics

Here we present a procedure to improve the error bound of the asymptotic solution to an arbitrary accuracy. All-order-asymptotics has appeared in [8]. There, the improved asymptotics are modified by a right multiplication to the strong asymptotics. This right multiplication can render the jump matrix become as close to the identity as one wishes (since the jump also acts to the right) by a certain well-defined recursive procedure, hence enabling one to get a solution with any improved error bound.

Our method is a little different in that we use a *left* multiplication on both outer parametrix and local parametrix. In the main text, we have already used such method without which the asymptotics would have failed at any half-integer \varkappa . Here we explain how this method can be applied to achieve an arbitrarily high order of accuracy in the error committed when matching the outer parametrix to the local parametrix at the hard-edge. It should be said that there is no practical purpose in improving the error that comes from the hard-edge (or outpost) parametrix to better than $\mathcal{O}(N^{-1})$ since then errors coming in the matching between the local parametrices at the other (simple) turning points are already of that order. In principle the same idea could be applied to the endpoints, but the details become different and in the interest of conciseness we refrain from such analysis.

Inside the disk \mathbb{D} , the strong asymptotics of \tilde{Y} is the product of the global parametrix and the local parametrix, i.e. $\tilde{Y}_\infty = \Psi_\varkappa R_\varkappa$. Outside the disk \mathbb{D} the strong asymptotics is simply $\tilde{Y}_\infty = \Psi_\varkappa$ and, therefore, we have a jump R_\varkappa on the boundary $\partial\mathbb{D}$.

To have a good asymptotic solution, this jump must be as close to the identity as possible. To this effect we will “improve” R_\varkappa ⁶ by a suitable left multiplication. This left multiplication will introduce a new singularity. Since the strong asymptotics $\Psi_\varkappa R_\varkappa$ must be holomorphic at the outpost, Ψ_\varkappa also needs to be modified by a *left* multiplication (that resembles a Schlesinger transformation) so as to cancel the singularity. Below we describe the whole procedure.

Using (3.32) $R_K := \tilde{z}^{-K\sigma_3} H_K = \zeta^{-K\sigma_3} (\mathbf{1} + \mathcal{O}(1/\zeta)) \zeta^{K\sigma_3}$ we can obtain the following expansion.

$$\tilde{z}^{K\sigma_3} R_K \tilde{z}^{-K\sigma_3} = \mathbf{1} + \frac{Y_1}{\tilde{z}} + \frac{Y_2}{\tilde{z}^2} + \cdots, \quad (\text{B.1})$$

where Y_j ’s are matrices of order $\mathcal{O}(N^{-\gamma_j})$ from the consideration of $\tilde{z} \sim N^{-\gamma}\zeta$. This grading property is crucial for the error analysis.

Since $\det H_K \equiv 1$ both sides of the above equation (B.2) are also uni-modular. In such case, we claim that the following factorization can be done

$$\tilde{z}^{K\sigma_3} R_K \tilde{z}^{-K\sigma_3} = \left(\mathbf{1} + \frac{M_1}{\tilde{z}} \right) \left(\mathbf{1} + \frac{\tilde{M}_1}{\tilde{z}} \right) \left(\mathbf{1} + \frac{M_2}{\tilde{z}^2} \right) \left(\mathbf{1} + \frac{\tilde{M}_2}{\tilde{z}^2} \right) \cdots \left(\mathbf{1} + \frac{M_p}{\tilde{z}^p} \right) \left(\mathbf{1} + \frac{\tilde{M}_p}{\tilde{z}^p} \right) \times \quad (\text{B.2})$$

$$\times \left(\mathbf{1} + \mathcal{O}\left(\frac{1}{\tilde{z}^{p+1}}\right) \right) \quad \text{for arbitrarily large positive integer } p, \quad (\text{B.3})$$

where M_j s (\tilde{M}_j s) are all nilpotent matrices so that each factor is uni-modular.

The proof proceeds by induction. First, any uni-modular matrix that has an expansion $\mathbf{1} + \frac{A}{\tilde{z}^{p+1}} + \mathcal{O}\left(\frac{1}{\tilde{z}^{p+2}}\right)$ must have a traceless matrix A . Secondly, any 2 by 2 traceless matrix can be represented as the sum of two nilpotent

⁶We will use the same notations R_\varkappa for the “improved” local parametrix and Ψ_\varkappa for the “improved” global parametrix.

matrices, say $A = M_{p+1} + \widetilde{M}_{p+1}$. Then the series can be expressed as

$$\mathbf{1} + \frac{A}{\widetilde{z}^{p+1}} + \mathcal{O}\left(\frac{1}{\widetilde{z}^{p+2}}\right) = \left(\mathbf{1} + \frac{M_{p+1}}{\widetilde{z}^{p+1}}\right) \left(\mathbf{1} + \frac{\widetilde{M}_{p+1}}{\widetilde{z}^{p+1}}\right) \left(\mathbf{1} + \mathcal{O}\left(\frac{1}{\widetilde{z}^{p+2}}\right)\right). \quad (\text{B.4})$$

This gives the recursive step of increasing p by one in (B.2). **Q.E.D.**

With such expansion in mind, we define the improved local parametrix by

$$R_{\varkappa} := \widetilde{z}^{-K\sigma_3} \left(\mathbf{1} - \frac{M_1}{\widetilde{z}}\right) \left(\mathbf{1} - \frac{\widetilde{M}_1}{\widetilde{z}}\right) \cdots \left(\mathbf{1} - \frac{M_p}{\widetilde{z}^p}\right) \left(\mathbf{1} - \frac{\widetilde{M}_p}{\widetilde{z}^p}\right) \widetilde{z}^{K\sigma_3} R_K. \quad (\text{B.5})$$

Here we have used $\left(\mathbf{1} + \frac{M}{\widetilde{z}}\right)^{-1} = \mathbf{1} - \frac{M}{\widetilde{z}}$ for any nilpotent M . Using the grading property that we mentioned earlier, we obtain $R_{\varkappa} = \mathbf{1} + \mathcal{O}(N^{-\gamma(p+1)})$, which will be the error bound of our improved asymptotics.

Now we improve the global parametrix such that $\Psi_{\varkappa} R_{\varkappa}$ is holomorphic inside \mathbb{D} . We recall that Ψ_K has a singularity at the outpost such that $\Psi_K \widetilde{z}^{-K\sigma_3}$ is holomorphic at the outpost. We propose that

$$\Psi_{\varkappa} = F(z) \Psi_K = \widetilde{F}_p(z) F_p(z) \cdots \widetilde{F}_2(z) F_2(z) \widetilde{F}_1(z) F_1(z) \Psi_K, \quad (\text{B.6})$$

$$F_j(z) := \mathbf{1} + \frac{F_{j1}}{z} + \frac{F_{j2}}{z^2} + \cdots + \frac{F_{jj}}{z^j} \quad (\text{and similarly for } \widetilde{F}_j(z)). \quad (\text{B.7})$$

The (recursive) construction follows. To illustrate the method let us write out the asymptotics.

$$\Psi_{\varkappa} R_{\varkappa} = \cdots \times \overbrace{\left(\mathbf{1} + \frac{\widetilde{F}_{11}}{z}\right) \left(\mathbf{1} + \frac{F_{11}}{z}\right) \Psi_K \widetilde{z}^{-K\sigma_3} \left(\mathbf{1} - \frac{M_1}{\widetilde{z}}\right) \left(\mathbf{1} + \frac{\widetilde{M}_1}{\widetilde{z}}\right)}^{\text{part 2}} \times \cdots. \quad (\text{B.8})$$

part 1

We understand $\Psi_K \widetilde{z}^{-K\sigma_3}$ is holomorphic at the outpost. First we will find F_{11} that makes “part 1” holomorphic at the outpost. Having a holomorphic “part 1”, we will then find \widetilde{F}_{11} that makes “part 2” holomorphic.

Obviously, these steps are recursive and we only need to clarify one generic step: finding j matrices, F_{j1}, \dots, F_{jj} , that satisfy the following analyticity condition.

$$\left(\mathbf{1} + \frac{F_{j1}}{z} + \cdots + \frac{F_{jj}}{z^j}\right) \mathbf{A}(z) \left(\mathbf{1} - \frac{M_j}{\widetilde{z}^j}\right) = \mathcal{O}(z^0), \quad (\text{B.9})$$

where $\mathbf{A}(z)$ is a matrix whose entries are analytic at the hard-edge (or outpost) which is determined from the previous step in the induction process (e.g. part 1 and part 2 in (B.8)). Denoting the expansion of \mathbf{A} as $\mathbf{A}(z) = A_0 + A_1 z + A_2 z^2 + \cdots$, and defining a new expansion:

$$\mathbf{A}^{[j]} := \mathbf{A}(z) \left(\frac{z}{\widetilde{z}}\right)^j = A_0^{[j]} + A_1^{[j]} z + A_2^{[j]} z^2 + \cdots, \quad (\text{B.10})$$

we can write the following solution:

$$(F_{j1}, \dots, F_{jj}) = \quad (\text{B.11})$$

$$= (A_0^{[j]}, \dots, A_{j-1}^{[j]}) \cdot M_j \cdot \left(\left[\begin{array}{cccc} 0 & \cdots & 0 & A_0 \\ 0 & \cdots & A_0 & A_1 \\ \vdots & & & \vdots \\ A_0 & A_1 & \cdots & A_{j-1} \end{array} \right] - \left[\begin{array}{cccc} A_1^{[j]} & A_2^{[j]} & \cdots & A_j^{[j]} \\ A_2^{[j]} & A_3^{[j]} & \cdots & A_{j+1}^{[j]} \\ \vdots & & & \vdots \\ A_j^{[j]} & A_{j+1}^{[j]} & \cdots & A_{2j-1}^{[j]} \end{array} \right] \cdot M_j \right)^{-1}. \quad (\text{B.12})$$

The global parametrix Ψ_{\varkappa} (B.6) is then fully determined by the recursive procedure.

C Construction of the outer parametrix for arbitrary number of cuts

We refer to the corresponding appendix in [1] since the formulæ are identical, the only difference being in the picture that accompanies the appendix.

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